A Problem suggested in the Geometry of Nets of Curves and applied to the Theory of Six Points having multiply Perspective Relations.

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1. A 1.1 correspondence between the curves of two nets of curves, one in a plane Ξ, the other in a plane Π, may be established by making four curves of one net no three of which are linearly related, correspond to four such curves of the other net. Let two arbitrary curves of the net in $\frac{\Xi}{\Pi}$ intersect, besides in the common base point system (if any), in a group of $\frac{p}{q}$ points; the 1.1 correspondence between the curves of the nets establishes a $p.q$ correspondence between the points of the two planes Ξ, Π.

In particular, for $p = q = 1$, there is a 1.1 correspondence between the points of the planes, *i. e.* a Cremona transformation changes one plane of points into the other. The geometry of the net of curves and the points* of Ξ is transformed into a similar geometry of the net of curves and the points of Π. One and so simultaneously both of the nets of curves may be set in correspondence with the net of lines in a plane Ω, and hence also arises a Cremona transformation between the points of Ω and Ξ, and of Ω and Π. Thus it is natural to speak of the geometry of the net of curves and the points of Ξ or Π in the terms adopted in the (ordinary) geometry of the lines and points of a plane Ω.

In the plane Ω let a figure E be determined by certain points; these determining points of E are subject to certain conditions or limitations as to generality of position in order that the figure E may enjoy certain properties. Assuming as fixed the correspondence between the curves of Π and the lines of Ω, the

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* That is, geometric constructions and theorems which are or *may be* completely expressed in terms of points and curves of the net.

corresponding figure F of Π is determined by the corresponding points, which are subject to the corresponding conditions (which now involve the determining points of F separately and these points together with the base point system of the net in Π) in order that the figure F may enjoy the corresponding properties.

However, after the general character of the net in Π, that is, the order of the curves and the full nature of the base point, or say *principal*, system has been determined upon, this principal system depends still upon certain arbitrary constants or parameters connected with the determination of position in Π. Now considering these parameters of the principal system of Π at our disposal, it will in general be possible, by the suitable determination of an equal number of parameters, to relieve the determining points of F from all the conditions which involve also the principal system; this leaves the points of F subject to say *internal* conditions only.

Then, when the points of F have been fixed subject to the remaining internal conditions, a principal system (definite or with certain parameters still remaining equal in number to the excess of the original number of parameters over the number of conditions from which the points of F have been relieved) may be found; that is, a net of curves in Π of the required nature may be found, so that in the geometry of that net of curves the points of F shall determine a figure F having the required properties corresponding to the properties of the figure E in the plane Ω.

It is possible that the points of F, in particular positions, satisfy not only the required internal conditions, but also others likewise internal, and such as to give to the principal system an equal number of parameters. (See §3.)

Especially noteworthy (*e. g.*, § 3, end, §4 fg.) are the cases in which in this way the points of F are released entirely from conditions. This will be possible only when the number of conditions imposed upon them for a fixed principal system in Π is equal to or less than the number of parameters of the principal system. (The converse is not true; see §3, the principal system has six parameters, for $n = 7$ there are only five conditions, but of these two are strictly *internal* and cannot be relieved by suitable choice of principal system.)

2. In the following paper I wish, when the net of curves under consideration in Π is the net of conics through three points, to make this determination of the principal system for two figures F; in the second case the figure will receive further consideration on its own account.

Consider between the points of the two planes Ω, Π a quadratic transformation with as principal system in each plane three distinct non-collinear points, say in Π, $H_1H_2H_3$. Then with well-understood exceptions relating to the principal system in Ω, in the plane Π to a point in Ω corresponds a point, to a line in Ω, a conic passing through the three points H_ι, and to the net of lines in Ω, the net of conics through the three points H_ι. The (ordinary) geometry of the points and lines of Ω is transformed into the geometry of the points and this net of conics in the plane Π, say for brevity into the geometry of the plane $\Pi_{C^2(H_1H_2H_3)}$, where the subscript indicates the net of curves in Π corresponding to the net of lines in Ω; or, where no ambiguity will arise, the geometry of the plane $\Pi_{H_1H_2H_3}$ (indicating merely the principal points of the net) or even of the plane Π_H. The principal system consisting of three points has six parameters.

Case I. 3. The simplest possible case will be taken as a good illustration.

In Ω let the figure E consist of n points ($n > 2$) subject to the $n - 2$ conditions that they shall lie in the same straight line.

In Π_H likewise the n points of F will be subject to the $n - 2$ conditions that they shall all lie on a conic passing through $H_1H_2H_3$.

First, $n > 5$. Then these $n - 2$ conditions are equivalent to the $n - 5$ *internal* conditions that they shall all lie on the conic determined by any five of them. and the three conditions that this conic shall pass through H_1, H_2, and H_3. Let n points $P_\kappa\ (\kappa = 1 \ldots . n)$ be subject merely to the $n - 5$ internal conditions that they shall be conconical* on say conic C^2. We expect to find a principal system H with $(6 - 3 =)\ 3$ remaining parameters, so that the n points shall be conconical with the points H_ι; and in fact each point H_ι must lie on C^2, but has on it one degree of freedom or its determination on it involves one parametric constant. Here the three conditions imposed upon the principal system are three independent conditions, one on each point; and so the three parameters remaining are independent, one for each point. We cannot, for instance, take H_1 arbitrarily (using two parameters) and afterwards determine H_2H_3 (with still one parameter).†

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* *Conconical;* just as any number of points lying on $\left\{\begin{matrix}\text{a line} \ldots\ldots\ldots\ldots\ldots\ldots \\ \text{the circumference of a circle}\end{matrix}\right\}$ are said to be $\left\{\begin{matrix}\textit{collinear} \\ \textit{concyclic}\end{matrix}\right\}$.

† Compare §12, where the principal system has four remaining parameters, which may be used in determing H_1H_2 at will, where H_3 is determined as one of a finite number (two) of points.

Exceptional cases arise when no five of the n points fully determine a conic:

(a) If $n-1$ of the n points say P_ι $(\iota=1\ldots.n-1)$ are collinear, say on the line p, the conic is p together with any line of the pencil through the remaining point P_n

(b) If the n points are collinear on p, the conic is p together with any line of the net of lines in the plane Π.

These afford illustrations of the remark near the close of §1. That is, in

(a) $n-3$ internal conditions are satisfied by the n points, two more than the required $n-5$, which may be interpreted as the conditions that the conic C^2 should decompose into two lines and then that it should be any conic of a certain pencil, that is, should have one parameter; of these two additional internal conditions only the latter relieves the principal system H so that it has $(3+1=)\,4$ parameters; say take H_1 arbitrarily, and H_2H_3 each arbitrarily on the line-pair (P_nH_1, p): while in

(b) $n-2$ internal conditions are satisfied by the n points, three additional, of which only the two that the (degenerate-) conic should be any conic of a net, *i. e.* should have two parameters, relieve the principal system H, so that it has $(3+2=)\,5$ parameters; say take H_1, H_2 arbitrarily, then H_3 arbitrarily on the line-pair (H_1H_2, p).

Second, $n \overline{\gtrless} 5$. Here the n points are subject to no internal conditions, but may be taken at random. There are then the $n-2$ conditions on the principal system H that the three points H_ι shall be conconical with the n points P_κ; the principal system subject to these conditions has $(6-(n-2)=)\,8-n$ parameters remaining.

Case II. §§4–11.

4. Let the figure E in Ω be two triangles in perspective from a certain centre, *i. e.* for a certain correspondence between the vertices of the triangles, the three lines joining corresponding vertices are concurrent in a point, the centre of perspective. The points of the figure satisfy one condition.

In Π_H the corresponding figure F consists of two triangles $A_1A_2A_3$, $B_1B_2B_3$, such that the conics C_ι^2 for $\iota=1, 2, 3$ pass through a fourth common point I, where C_ι^2 is the conic through the five points $H_1H_2H_3A_\iota B_\iota$. We say, in the plane Π_H the two triangles are in perspective from the centre I. Stated otherwise, there are three conics C_ι^2 of the pencil circumscribing the 4-gon $H_1H_2H_3I$ such that C_ι^2 passes through the points $A_\iota B_\iota$ $(\iota=1, 2, 3)$.

Taking the two triangles, however, at random, the principal system H is subject to one condition and has five parameters remaining; that is, taking H_1H_2 at random, H_3 must lie on a certain locus, and for a definite position of H_3 on this locus the centre I will be determined uniquely; the second way of looking at the figure shows that I lies on the same locus and has H_3 for its corresponding centre. The two points H_3, I play entirely the same rôle and so may better be written G_1, G_2.

Then given H_1, H_2, A_ι, B_ι $(\iota=1,2,3)$ at random, we are required to determine G_1, G_2 so that certain three conics $C_\iota^2(H_1H_2A_\iota B_\iota)$ shall pass through G_1, G_2 forming a pencil with the base-points $H_1H_2G_1G_2$.

5. Two quadrangles have circumscribing them two pencils of conics, any two of which meet in a group of points. Thus is determined an involution of the points of the plane, *i. e.* every point belongs to one and only one such group. The points of a group are said to be conjugate to one another, or, to specify the involution, conjugate with respect to the two quadrangles. Every group consists of 4, 3, 2, 1 points (the last case is nugatory) according as the two quadrangles have 0, 1, 2, 3 vertices in common.

In this way the two quadrangles $(H_1H_2A_{\iota+1}B_{\iota+1})\,(H_1H_2A_{\iota+2}B_{\iota+2})$* determine a quadratic involution of the points of the plane Π; to any point X corresponds a point $Y_{\iota+1,\,\iota+2}$, the fourth point of intersection of the two conics $C^2_{\lambda,\,X}\equiv C^2(H_1H_2A_\lambda B_\lambda X)$, $(\lambda=\iota+1,\iota+2)$.* The points X, $Y_{\iota+1,\,\iota+2}$ are conjugate with respect to the two quadrangles or say with respect to $A_{\iota+1}B_{\iota+1}$, $A_{\iota+2}B_{\iota+2}$.

Clearly the points G_1G_2 are conjugate with respect to $A_{\iota+1}B_{\iota+1}$, $A_{\iota+2}B_{\iota+2}$ for $\iota=1,2,3$, because $C^2_{\lambda,\,G_1}$ passes through G_2 $(\lambda=1,2,3)$; conversely, two points in these three ways conjugate are points G_1G_2 (except points XY such that $C^2_{1,\,X}$, $C^2_{2,\,X}$, $C^3_{3,\,X}$ having H_1H_2XY in common yet do not form a pencil; that is, have ∞ points in common and so break up into a common line and three non-concurrent lines; this case arises for any two points XY on H_1H_2 unless A_1B_1, A_2B_2, A_3B_3 are concurrent).

If two points XY are in two of these three ways conjugate, since then $C^2_{\lambda,X}$ for $\lambda=1,2,3$ must pass through Y, they are also in the third way conjugate.

6. We find the locus of the points G_1G_2, that is, of the points XY at the

* Throughout the paper the subscripts depending on ι are to be taken *modulo* 3.

same time conjugate with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3, or say of the points X for which Y_{12} and Y_{13} coincide, $Y_{12} \equiv Y_{13} \equiv Y$.

As X describes the range of points lying on a line l passing through H_2, the conic $C^2_{\iota, X} \equiv C^2(H_1H_2A_\iota B_\iota X)$ describes a pencil of conics circumscribing the quadrangle $H_1H_2A_\iota B_\iota$, which (since l passes through H_2, a base-point of the pencil) is projective with the range l; and hence for $\iota = 1, 2, 3$ the three pencils of conics are projective with one another.

Any two corresponding conics $C^2_{1, X}$, $C^2_{2, X}$ intersect in (H_1, H_2 and) two points X, Y_{12} conjugate with respect to A_1B_1, A_2B_2. The locus of the (variable) points of intersection of corresponding conics of the two projective pencils $C^2_{1, X}(H_1H_2A_1B_1)$, $C^2_{2, X}(H_1H_2A_2B_2)$, is a curve of the fourth order having double points at H_1H_2 and passing through A_1, B_1, A_2, B_2; which here consists of the line l, the locus of the point of intersection X and a certain cubic $C^3_{12, l}(H_1^2H_2A_1B_1A_2B_2)$ having a double point at H_1 and passing through the other five points as indicated by the parenthesis, the locus of the other point of intersection Y_{12}. In like manner there is a cubic $C^3_{13, l}(H_1^2H_2A_1B_1A_3B_3)$ the locus of points Y_{13} conjugate to the points X of l.

Let the ray l describe the pencil about H_2; its corresponding $C^3_{12, l}$ will then describe a pencil of cubics whose nine base points are evident, H_1 counting for four; a line through H_2 has one corresponding cubic of this pencil; conversely, a cubic has one corresponding line, since any point on it U quâ Y_{12} by its conjugate point X determines a line $H_2X \equiv l$, and the cubic C^3_{12, H_2X} corresponding to this line H_2X must be (the) one of the pencil passing through U, that is, the original cubic on which U lay. Thus the pencil of rays l through H_2 is projective with the pencil of cubics $C^3_{12}(H_1^2H_2A_1B_1A_2B_2)$, and likewise with the pencil $C^3_{13}(H_1^2H_2A_1B_1A_3B_3)$.

If for a point X, $Y_{12} \equiv Y_{13} \equiv Y$, setting $H_2X \equiv l$, Y quâ Y_{12} lies on $C^3_{12, l}$, Y quâ Y_{13} lies on $C^3_{13, l}$, and so Y is a point of intersection of two corresponding cubics of the pencils C^3_{12}, C^3_{13}. Hence the locus of points XY conjugate at once with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3 is contained in the locus of points of intersection of corresponding cubics of the two projective pencils C^3_{12}, C^3_{13}.

7. This latter locus is a curve of the sixth order, $C^6_1(H_1^4H_2^2A_1^2B_1^2A_2B_2A_2B_3)$. It contains the two lines H_1A_1, H_1B_1. For let X describe the line H_2A_1. Then for X not H_2 or A_1, $C^2_{1, X} \equiv H_2A_1 . H_1B_1$ and $C^2_{2, X} \equiv C^2(H_1H_2A_2B_2X)$, which intersect at X on H_2A_1 and at Y_{12} on H_1B_1. Thus as X describes H_2A_1, Y_{12}

describes H_1B_1 projectively. In fact the cubic of pencil $C_{12}^3(H_1^2H_2A_1B_1A_2B_2)$ which contains one other point of H_1B_1 breaks up into this line H_1B_1 and the conic $C_{2,A_1}^2 \equiv C^2(H_1H_2A_2B_2A_1)$; (this conic is locus of points Y_{12} conjugate to the ∞^1 points X adjacent to and surrounding the base-point A_1).

Thus for $l_{(1)} \equiv H_2A_1$,

$$C_{12,(1)}^3 \equiv \overline{H_1B_1}\,.\,C_{2,A_1}^2,$$

and likewise

$$C_{13,(1)}^3 \equiv \overline{H_1B_1}\,.\,C_{3,A_1}^2;$$

and also for $l_{(2)} \equiv H_2B_1$,

$$C_{12,(2)}^3 \equiv \overline{H_1A_1}\,.\,C_{2,B_1}^2,$$

$$C_{13,(2)}^3 \equiv \overline{H_1A_1}\,.\,C_{3,B_1}^2.$$

The equation of this locus is then by suitable choice of constant factors, writing $C^3 = 0$ for equation of a curve C^3,

$$\frac{C_{12,(1)}^3}{C_{12,(2)}^3} = \frac{C_{13,(1)}^3}{C_{13,(2)}^3};$$

that is,

$$\frac{\overline{H_1B_1}\,.\,C_{2,A_1}^2}{\overline{H_1A_1}\,.\,C_{2,B_1}^2} = \frac{\overline{H_1B_1}\,.\,C_{3,A_1}^2}{\overline{H_1A_1}\,.\,C_{3,B_1}^2},$$

or

$$\overline{H_1A_1}\,.\,\overline{H_1B_1}\,(C_{2,A_1}^2C_{3,B_1}^2 - C_{2,B_1}^2C_{3,A_1}^2) = 0;$$

say the locus is

$$C_1^6(H_1^4H_2^2A_1^2B_1^2A_2B_2A_3B_3) \equiv \overline{H_1A_1}\,.\,\overline{H_1B_1}\,.\,C_1^4(H_1^2H_2^2A_1B_1A_2B_2A_3B_3)$$

where

$$C_1^4 \equiv C_{2,A_1}^2C_{3,B_1}^2 - C_{2,B_1}^2C_{3,A_1}^2.$$

The locus of points XY conjugate in the three ways is contained in each the three loci of the points of intersection of corresponding cubics for the th pairs of projective pencils (C_{12}^3, C_{13}^3), (C_{12}^3, C_{23}^3), (C_{13}^3, C_{23}^3). The first as just proved contains the two lines $\overline{H_1A_1}.\overline{H_1B_1}$ and the other two in like manner contain $(\overline{H_1A_2}.\overline{H_1B_2})$, $(\overline{H_1A_3}.\overline{H_1B_3})$ respectively. These lines being in general *distinct*, do not belong to the locus of points XY conjugate in the three ways, which latter is hence a quartic $C^4(H_1^2H_2^2A_1B_1A_2B_2A_3B_3)$, whose equation may be written in three ways,

$$C^4 \equiv C_{\iota+1,A_\iota}^2C_{\iota+2,B_\iota}^2 - C_{\iota+1,B_\iota}^2C_{\iota+2,A_\iota}^2 = 0 \qquad (\iota=1,2,3),$$

putting in evidence that it passes also through the six points D_ι, E_ι, defined as the fourth points of intersection of the two conics

$$(C_{\iota+1,A_\iota}^2,\ C_{\iota+2,A_\iota}^2),\ (C_{\iota+1,B_\iota}^2,\ C_{\iota+2,B_\iota}^3) \text{ respectively.}$$

Here it has, however, been assumed that C_1^6 contains no extraneous factors besides the lines H_1A_1, H_1B_1. This appears as follows: Let Y' be a point of intersection of two corresponding cubics $C^3_{12,l}$, $C^3_{13,l}$. To Y' quâ $\left\{\begin{matrix} Y_{12} \text{ on } C^3_{12,l} \\ Y_{13} \text{ on } C^3_{13,l} \end{matrix}\right.$ corresponds a point $\left\{\begin{matrix} X_{12} \\ X_{13} \end{matrix}\right.$ common to $\left\{\begin{matrix} l,\ C^2_{1,Y'},\ C^2_{2,Y'} \\ l,\ C^2_{1,Y'},\ C^2_{3,Y'} \end{matrix}\right.$. Now $C^2_{1,Y'}(H_1H_2A_1B_1Y')$ has already H_2 in common with l, and hence either (1) $C^2_{1,Y'}$ intersects l in only one other point $X_{12}\equiv X_{13}$, in which case the point of intersection Y' of two corresponding cubics is really one of a pair of points Y', $X_{12}\equiv X_{13}$ conjugate at once with respect to A_1B_1, A_2B_2 and to A_1B_1, A_3B_3; or (2) $C^2_{1,Y'}$ contains the ray l completely (*i. e.* $C^2_{1,Y'}$ degenerates into a line-pair), in which case we do do not at all know that X_{12} common to l and $C^2_{2,Y'}$ coincides with X_{13} common to l and $C^2_{3,Y'}$, and in fact in general it does not.

Hence if the two corresponding cubics do intersect in a point Y' not one of a pair of points in the two ways conjugate, they must correspond to a line l which a conic of pencil C_1^2 contains entirely, *i. e.* $l\equiv H_2A_1$, H_2B_1 or H_2H_1. The cubics corresponding to $H_2A_1\,(H_2B_1)$ have been shown to have in common the extraneous factor $H_1B_1\,(H_1A_1)$. The cubics corresponding to H_1H_2 must pass through the intersection of A_1B_1, A_2B_2 and of A_1B_1, A_3B_3 respectively; they do not coincide and so have in common no extraneous factor. The assumption referred to is thus justified.

8. Thus the locus of the points G_1G_2 such that the conics C^2_{ι,G_1} ($\iota=1,\ 2,\ 3$) form a pencil with the base-points $H_1H_2G_1G_2$ is a binodal quartic

$$\mathbf{C}^4\equiv C^4(H_1^2H_2^2A_\iota B_\iota D_\iota E_\iota) \qquad ({\scriptstyle \iota=1,2,3}),$$

on which the point-pairs G_1G_2 form a quadratic involution.

In case $H_1H_2A_1B_1A_2B_2$ are conconical, *i. e.* $C^2_{2,A_1}\equiv C^2_{2,B_1}$, then the locus quartic $\mathbf{C}^4$ contains this conic say $\mathbf{C}^2_{12}$, which is divided in an involution of points G_1G_2 by the pencil of conics C_3^2.

If further $H_1H_2A_1B_1A_3B_3$ are conconical on $\mathbf{C}^2_{13}$, this is likewise a factor of the locus quartic $\mathbf{C}^4$, and $\mathbf{C}^4\equiv\mathbf{C}^2_{12}\mathbf{C}^2_{13}$.

If still further $H_1H_2A_2B_2A_3B_3$ are conconical on $\mathbf{C}^2_{12}$, this is in like manner a factor of $\mathbf{C}^4$; this can be if $\mathbf{C}^2_{23}\equiv\mathbf{C}^2_{31}\equiv\mathbf{C}^2_{12}\equiv$ say $\mathbf{C}^2_{123}$, *i. e.* if the eight points $H_1H_2A_\iota B_\iota$ (${\scriptstyle \iota=1,2,3}$) are conconical, or if in any way by the breaking up of the conics $\mathbf{C}^2_{12}\mathbf{C}^2_{13}$ contains $\mathbf{C}^2_{23}$ as a factor; unless this is so, $\mathbf{C}^4$ must be indeterminate.

These special cases may well be considered independently and more fully.

9. In the plane Π let three (non-coincident) conics

$$\mathbf{C}^{2\prime}_{\iota+1,\,\iota+2}(H_1H_2A'_{\iota+1}B'_{\iota+1}A'_{\iota+2}B'_{\iota+2}) \qquad (\iota=1,2,3)$$

have a common chord H_1H_2, then their three other chords $A'_\iota B'_\iota$ $(\iota=1,2,3)$ are concurrent* say at G'_2.

Transform the plane into itself by a quadratic transformation with $H_1H_2G_1$ (where G_1 is chosen at random in the plane) as principal points. The resulting theorem may be expressed thus: Let three (non-coincident) conics

$$\mathbf{C}^{2}_{\iota+1,\,\iota+2}(H_1H_2A_{\iota+1}B_{\iota+1}A_{\iota+2}B_{\iota+2}) \qquad (\iota=1,2,3)$$

exist. Then taking any point G_1 of the plane, the three conics $C^2_{\iota,\,G_1}$ are concurrent at G_2 forming a pencil.

Thus is established a quadratic involution of the points G_1G_2 of the plane, which must be identical with the quadratic involution of points conjugate with regard to A_1B_1, A_2B_2 conconical with H_1H_2 on $\mathbf{C}^2_{12}$. ($\mathbf{C}^4$ is indeterminate.)

If H_1H_2 are the two circular points at infinity, the quadratic involution is that of inverse points with reference to the radical centre and the orthogonal circle of the three circles. The known generalization of this *inversion* or *transformation by reciprocal radii vectores* is then to be considered, since by it these special cases become clear.

10. Given a point C and a conic C^2_C. Two points $R'R''$ are said to be *inverse* with respect to the centre C and the conic C^2_C (or briefly, with respect to C and C^2_C) when R' and R'' are collinear with C and are conjugate with respect to C^2_C, *i. e.* the segment $R'R''$ is divided harmonically by C^2_C.

Let H_1H_2 be the chord of contact of tangents through C to C^2_C. The points $W_1 \equiv H_1R'.H_2R''$, $W_2 \equiv H_1R''.H_2R'$ are called the *anti-points* of R' and R'' with respect to H_1H_2; R' and R'' being inverse with respect to C and C^2_C, the anti-points lie on C^2_C.† Hence the inverse of a line H_1W where W is on C^2_C is the line H_2W. Any line through C of course inverts into itself. Points on H_1H_2, CH_1, CH_2 invert into C, H_1, H_2 respectively, or really into elements of direction

* Salmon's Conic Sections, 6th ed., §266.

† Let the line $CR'R''$ intersect H_1H_2 in U and C^2_C in V_1V_2. Then on this line CUV_1V_2 and $R'R''V_1V_2$ are two harmonic ranges. The harmonic pencil $H_1(CUV_1V_2)$ gives on the conic the harmonic range $H_1H_2V_1V_2$, H_1C being tangent at H_1. Let H_1R' intersect C^2_C in (H_1 and also) W. The pencil $W(H_1H_2V_1V_2)$ is harmonic and cuts $CR'R''$ in the harmonic range $R'(WH_2.R'R'')V_1V_2$, where since $R'R''V_1V_2$ is harmonic, $WH_2.R'R'' \equiv R''$, *i. e.* WH_2R'' are collinear; ∴ in fact H_1R' and H_1R'' intersect at (W or) W_1 *on the conic*, and likewise for W_2.

at those points. This inversion is then a quadratic transformation of the plane with CH_1H_2, CH_2H_1 as corresponding principal points. A conic through H_1H_2 inverts into a conic through H_1H_2. In particular, a conic through two inverse points $R'R''$ intersecting C_C^2 in T and U, $C^2(H_1H_2R'R''TU)$ inverts into a $C^2(H_1H_2R''R'TU)$, *i. e.* into itself; any two points $S'S''$ on it collinear with C are inverse; in particular, CT is tangent to the conic at T, which may be expressed by the theorem that C_C^2 is the locus of the points of tangency of tangents through C to all conics (which invert into themselves) of a pencil through a pair of inverse points and H_1H_2. Any two pairs of inverse points are conconical with H_1H_2.

Conversely, given any six conconical points $H_1H_2A_1B_1A_2B_2$ on $\mathbf{C}_{12}^2$, take $C \equiv A_1B_1 . A_2B_2$ and C_C^2 that conic of pencil tangent to CH_1, CH_2 at H_1, H_2 which divides A_1B_1 harmonically; then A_1B_1 are inverse with respect to C and C_C^2 and, by what precedes, also A_2B_2 are inverse.

Two conics C_1^2, C_2^2 each through H_1H_2 and a pair of inverse points A_1B_1, A_2B_2 intersect in two points X_1Y_{12}, and since each conic inverts into itself, these points are a pair of inverse points.* That is, any two points XY_{12} conjugate with respect to A_1B_1, A_2B_2 in the sense of §5, *where* $H_1H_2A_1B_1A_2B_2$ *are conconical on* $\mathbf{C}_{12}^2$, are inverse points in the inversion determined by H_1H_2 as base-points and by A_1B_1, A_2B_2 as two pairs of inverse points; and conversely. Notice, however, as exception, that any point on $\mathbf{C}_{12}^2$ is with respect to A_1B_1, A_2B_2 conjugate to every other one.

11. Let $H_1H_2A_1B_1A_2B_2$ be conconical on $\mathbf{C}_{12}^2$, thus determining a certain centre C and conic C_C^2 of an inversion. Then the binodal quartic $\mathbf{C}^4$ degenerates into $\mathbf{C}_{12}^2$ and a conic of pencil $C_3^2(H_1H_2A_3B_3)$. The points conjugate with respect to A_1B_1, A_2B_2 are by §10 (except the point-pairs lying on $\mathbf{C}_{12}^2$) inverse. This conic of pencil C_3^2 quâ locus of point-pairs G_1G_2 conjugate in three ways must invert into itself; that is, it is that conic of pencil which joins the two pairs of inverse points A_3A_3', B_3B_3', say $C^2(H_1H_2A_3B_3A_3'B_3')$.†

This conic is determinate unless A_3B_3 are themselves inverse points. In fact in this case $\mathbf{C}^4$ is indeterminate, for $A_\iota B_\iota$ $(\iota=1, 2, 3)$ being inverse points,

* Whence easily a proof of the theorem referred to in §9.

† Observe that $\begin{cases} A_3' \\ B_3' \end{cases}$ may be defined as fourth intersection of $\begin{cases} C_{1,A_3}^2, C_{2,A_3}^2 \\ C_{1,B_3}^2, C_{2,B_3}^2 \end{cases}$; that is, A_3', $B_3' \equiv D_3$, E_3, in agreement with theory of §7. Here since $C_{1,A_2}^2 \equiv C_{1,B_2}^2 \equiv C_{2,A}^2 \equiv C_{2,B}^2 \equiv \mathbf{C}_{12}^2$, $D_1E_1D_2E_2$ are indeterminate.

any pair of inverse points XY is conconical with $H_1H_2A_\iota B_\iota\ (\iota=1,2,3)$, and so is a pair of points G_1G_2. There are two cases of this kind; (a) the six points $A_\iota B_\iota$ are conconical with H_1H_2 on one conic $\mathbf{C}^2_{123}$, and the chords $A_\iota B_\iota$ are concurrent at C; (b) the six points are the points of intersection by pairs of three conics $\mathbf{C}^2_{12}$, $\mathbf{C}^2_{13}$, $\mathbf{C}^2_{23}$ having the common chord H_1H_2.

If A_3B_3 are not inverse but yet conconical with $H_1H_2A_1B_1$ on $\mathbf{C}^2_{13}$, then $\mathbf{C}^2_{13}$ quâ C^2_1 inverts into itself and so coincides with $C^2(H_1H_2A_3B_3A'_3B'_3)$, and the quartic $\mathbf{C}^4$ breaks up into the two conics $\mathbf{C}^2_{12}$, $\mathbf{C}^2_{13}$. On $\left\{\begin{matrix}\mathbf{C}^2_{12}\\ \mathbf{C}^2_{13}\end{matrix}\right.$ the point-pairs G_1G_2 are given as points of intersection with conics of the pencil $\left\{\begin{matrix}C^2_3\\ C^2_2\end{matrix}\right.$. Observe that $H_1H_2A_2B_2A_3B_3$ cannot be conconical, or A_3B_3 would be inverse, unless the *eight* points are conconical on $\mathbf{C}^2_{123}$ and the chords $A_\iota B_\iota$ are not concurrent; in this latter case $\mathbf{C}^4 \equiv \mathbf{C}^2_{123}\mathbf{C}^2_{123}$ and any point of $\mathbf{C}^2_{123}$ is in three ways conjugate with every other point.

Application of the results of Case II. §§12–18.

12. Given six points $K_1 \ldots . K_6$ grouped in two ways (1)(2) into triples of pairs, $A_{\iota(1)}B_{\iota(1)}$, $A_{\iota(2)}B_{\iota(2)}\ (\iota=1,2,3)$. Taking H_1H_2 at random, two definite quartics $\mathbf{C}^4_{(1)}$, $\mathbf{C}^4_{(2)}$ of the net* of quartics $C^4(H_1^2H_2^2K_1 \ldots . K_6)$ are the loci of point pairs $G_{1(1)}G_{2(1)}$, $G_{1(2)}G_{2(2)}$ for the two correspondences or groupings (1), (2). These two quartics intersect in the base-points and also in $(4^2-(2.2^2+6)=)$ two other points I_1I_2. Let the point-pairs G_1G_2 for the correspondence

$$\left\{\begin{matrix}(1)\\(2)\end{matrix}\right. \text{ be on } \left\{\begin{matrix}\mathbf{C}^4_{(1)} & I_1J_{1(1)} \text{ and } I_2J_{2(1)}\\ \mathbf{C}^4_{(2)} & I_1J_{1(2)} \text{ and } I_2J_{2(2)}\end{matrix}\right. .$$

Then the six points $K_1 \ldots . K_6$ have the perspective relations (1) and (2) in the plane $\Pi_{H_1H_2I_1}$ from the centres $J_{1(1)}$, $J_{1(2)}$; that is, the conics

$$\left\{\begin{matrix}C^2_{\iota(1)}(H_1H_2I_1A_{\iota(1)}B_{\iota(1)})\\ C^2_{\iota(2)}(H_1H_2I_1A_{\iota(2)}B_{\iota(2)})\end{matrix}\right. \text{ for } \iota=1,\ 2,\ 3 \text{ from a pencil through} \left\{\begin{matrix}J_{1(1)} \text{ on } \mathbf{C}^4_{(1)}\\ J_{1(2)} \text{ on } \mathbf{C}^4_{(2)}\end{matrix}\right. .$$

They have the same perspective relations in the plane $\Pi_{H_1H_2I_2}$ from the centres $J_{2(1)}J_{2(2)}$.

Thus, taking H_1H_2 at random, two and only two points I_1I_2 can be found such that in each of the planes $\Pi_{H_1H_2I_1}$, $\Pi_{H_1H_2I_2}$ the six points $K_1 \ldots . K_6$ shall

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* Net; for a quartic is fully determined by 14 conditions, and here there are $2.3+6=12$.

have the two perspective relations according to the given correspondences (1), (2).

13. In particular, separating the six points into two triangles $K_1K_2K_3$, $K_4K_5K_6$, there are three say *cyclic* correspondences,

$$\begin{pmatrix} K_1K_2K_3 \\ K_4K_5K_6 \end{pmatrix}, \quad \begin{pmatrix} K_1K_2K_3 \\ K_5K_6K_4 \end{pmatrix}, \quad \begin{pmatrix} K_1K_2K_3 \\ K_6K_4K_5 \end{pmatrix},$$

say correspondences (1)(2)(3).

In a plane the theorem holds; if two triangles are perspective according to two correspondences of a cyclic set of three, then they are also perspective according to the third correspondence of that set.* By a quadratic transformation this is shown to hold also in a plane $\Pi_{H_1H_2H_3}$.

Now these two triangles are in perspective in the two ways (1)(2) in each of the planes $\Pi_{H_1H_2I_1}$, $\Pi_{H_1H_2I_2}$; hence in each plane they must be in perspective also in the third way (3), and hence the corresponding locus quartic $\mathbf{C}^4_{(3)}$ must pass through I_1, I_2.

The three locus quartics $\mathbf{C}^4_{(1)}\mathbf{C}^4_{(2)}\mathbf{C}^4_{(3)}$ of point-pairs G_1G_2 for three cyclic correspondences (1)(2)(3) of the vertices of two arbitrary triangles $K_1K_2K_3$, $K_4K_5K_6$ form a pencil in the net of quartics $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

14. The six points $K_1 \ldots . K_6$ may be divided in 10 ways into a pair of triangles; the vertices of two triangles may be arranged in triples of pairs of corresponding vertices in 6 ways (arising from the six permutations of the three vertices of one of the triangles); any grouping of the six points into triples of pairs, $A_\iota B_\iota$, (§12) gives a correspondence between the vertices of 4 pairs of triangles $\begin{pmatrix} A_1A_2A_3, & A_1A_2B_3, & A_1B_2A_3, & A_1B_2B_3 \\ B_1B_2B_3, & B_1B_2A_3, & B_1A_2B_3, & B_1A_2A_3 \end{pmatrix}$. Thus the six points may be grouped in $\frac{10.6}{4} = 15$ ways into triples of pairs of points. Every such grouping has a corresponding locus quartic $\mathbf{C}^4$ belonging to the net $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

From §13 and the remark above it is clear that every such locus quartic belongs in four ways to a pencil of three such quartics. I hope at another time to discuss more fully this system of 15 quartics in the net $C^4(H_1^2H_2^2K_1 \ldots . K_6)$.

15. Suppose that for a grouping $A_\iota B_\iota$ of the six given points $K_1 \ldots . K_6$ the lines $A_\iota B_\iota$ ($\iota = 1, 2, 3$) are concurrent say at L; that is, that the corresponding

* Von Staudt, Geometrie der Lage, p. 125; Rosanes, p. 550, and Schröter, p. 555 of the Math. Annalen, II.

perspective relation exists *in the simple plane* Π. Then clearly taking H_1H_2 at random and any point X on H_1H_2, the perspective relation exists identically in the plane $\Pi_{H_1H_2X}$ or $\Pi_{C^2(H_1H_2X)}$ from the centre L, for a *line* $C^2(H_1H_2X)$ in the plane $\Pi_{C^2(H_1HX)}$ consists of H_1H_2X itself and a line of the simple plane Π, and in particular the conics $C^2_{\iota\ X}$ from a pencil consisting of the common line H_1H_2X and the pencil of lines $A_\iota B_\iota$ through L. Thus, if for a grouping of the points $K_1 \ldots . K_6$ into a triple of pairs, the perspective relation holds in the simple plane Π, then the locus quartic consists of the line H_1H_2 itself and the cubic $C^3(H_1H_2K_1 \ldots . K_6L)$, *i. e.* that cubic of the pencil through the eight points $H_1H_2K_1 \ldots . K_6$ which passes also through the centre of perspective; this cubic is the real locus of point-pairs G_1G_2 for this grouping (the point L alone corresponding to all points on H_1H_2, and conversely), and one may speak in this case of the locus cubic.

If several perspective relations of the points K exist in the simple plane Π (that is, if for several groupings the lines $A_\iota B_\iota$ are concurrent), since the locus cubics for these groupings all pass through a point say H_3, the ninth base-point of the pencil of cubics through $H_1H_2K_1 \ldots . K_6$, the same perspective relations exist in the plane $\Pi_{H_1H_2H_3}$. Or more compactly, *whatever perspective relations among the six points K exist in the simple plane Π exist likewise in the plane $\Pi_{H_1H_3H_3}$ or $\Pi_{C^2(H_1H_2H_3)}$, where $H_1H_2H_3$ with $K_1 \ldots . K_6$ make up the nine base-points of a pencil of cubics.*

The converse is true; for if a perspective relation exist in $\Pi_{H_1H_2H_3}$, the corresponding locus quartic must pass through H_3; now the pencil of quartics of the net $C^4(H_1^2H_2^2K_1 \ldots . K_6)$ passing through H_3 consists of the common line H_1H_2 and the pencil of cubics through the nine base-points; thus this particular locus quartic must contain the line H_1H_2, and the perspective relation must hold in the plane $\Pi_{H_1H_2X}$ (X being any point on H_1H_2), or, what is identically the same, in the simple plane Π.

16. I give two illustrative cases in which the theorems of §§15, 18 have application: (*a*) Two triangles may be in 1, 2, 3, 4 or 6 ways in perspective; this last case does not occur when both triangles are real, the simplest illustration being an equilateral triangle and the triangle with vertices at the two circular points at infinity and at the centre of the circle circumscribing the first triangle.*

* Multiply perspective triangles were first discussed by Rosanes, Schroter, Math. Annalen II; later by Kantor, Ueber die Configurationen (3, 3) mit den Indices 8, 9, etc.; Sitzungsber. d. Wien. Akad. II Abtheilung, 1881, LXXXIV, 915–932; and by Hess, Beitrage zur Theorie der mehrfach perspectiven Dreiecken und Tetraeder, Math. Ann. 1886, XXVIII, 167–260.

(*b*) Six points may form a *Clebsch's six-gon* with the property that two triangles formed in any way with these six points as vertices are four-ply perspective.* The simplest illustration is the six-gon whose six vertices are the five vertices of a regular pentagon and its centre as sixth vertex.

17. If we transform the plane Π into itself by a quadratic transformation with any three points $G_{(1)}$, $G_{(2)}$, $G_{(3)}$ as self-corresponding principal points, the net of conics $C^2(H_1H_2H_3)$ transforms into the net of trinodal quartics $C^4(G^2_{(1)1}G^2_{(1)2}G^2_{(1)3}G_{(0)1}G_{(0)2}G_{(0)3})$ (where H_ι transforms into $G_{(0)\iota}$); we may say that the plane $\Pi_{C^2(H_\iota)}$ transforms into the plane $\Pi_{C^4(G^2_{(1)\iota}G_{(0)\iota})}$.

The theorem of §15 was, whatever perspective relations hold for the six points $K_1 \ldots . K_6$ in the simple plane Π hold also in the plane $\Pi_{C^2(H_\iota)}$, where the nine points $H_\iota K_\kappa$† are the nine base-points of a pencil of cubics C^3; and conversely.

By a quadratic transformation with the arbitrary principal points $G_{(1)\iota}$, this theorem becomes: Whatever perspective relations hold for the six points $K_1' \ldots . K_6'$ in the plane $\Pi_{C^2(G_{(1)\iota})}$ hold likewise in the plane $\Pi_{C^4(G^2_{(1)\iota}G_{(0)\iota})}$ where the nine points $G_{(0)\iota}K_\kappa'$ are the 9 base-points of a pencil of $C^6(G^3_{(1)\iota})$; and conversely.

By mathematical induction, using successive quadratic transformations, the general theorem is proved:

Given six points $K_1 \ldots . K_6$ and $3(n+1)$ points $G_{(\lambda)\iota}\left(\begin{smallmatrix}\lambda=0,1,\ldots n\\ \iota=1,2,3\end{smallmatrix}\right)$ of which all are arbitrary except $G_{(0)3}$, which is determined (in general uniquely) by the condition that the nine points $G_{(0)\iota}K_\kappa$ shall be the 9 base-points of a pencil of $C^{3.2^n}\big(\underset{\lambda=n\ldots.1}{G^{3.2^{\lambda-1}}_{(\lambda)\iota}}\big)$;‡ then whatever perspective relations among the six points K hold in the plane

$$\Pi_{C^{2^n}\left(\underset{\lambda=n\ldots.1}{G^{2^\lambda-1}_{(\lambda)\iota}}\right)}$$

hold also in the plane

$$\Pi_{C^{2^{n+1}}\left(\underset{\lambda=n\ldots.1}{G^{2\lambda}_{(\lambda)\iota}G_{(0)\iota}}\right)} \equiv \Pi_{C^{2^{n+1}}\left(\underset{\lambda=n\ldots.0}{G^{2\lambda}_{(\lambda)\iota}}\right)},$$

and conversely.

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* This six-gon discovered by Clebsch, Math. Ann. 1871, IV, 284, was further discussed, from the standpoint of ikosaedron investigations, by Klein and by Hess, under the name "Das zehnfach Brianchon'sche Sechseck," and more fully by Schröter, "Das Clebsch'sche Sechseck," Math. Ann. 1887, XXVIII, 457–482, to whom the latter probably permanent name is due.

† Throughout this number and the following ones $\iota=1, 2, 3$, $\kappa=1, 2 \ldots . 6$.

‡ $3\sum_{\lambda=1}^{\lambda=n}(3.2^{\lambda-1})^2+9\equiv(3.2^n)^2$.

18. Given $K_1 \ldots . K_6$ with certain perspective relations in the simple plane Π. Define any number of successive triples of points $H_{(\lambda)\iota}$ as follows:

Let $K_K H_{(0)\iota}$ be the nine base-points of a pencil of C^3,

and then $K_K H_{(1)\iota}$ " " " " " $C^6(H^3_{(0)\iota})$,

and then $K_K H_{(2)\iota}$ " " " " " $C^{12}(H^6_{(0)\iota} H^3_{(1)\iota})$,

and so on; that is, in general, let

$K_K H_{(m)\iota}$ be the nine base-points of a pencil of $C^{3\cdot 2^m}\underset{\lambda=0.1\ldots.m-1}{(H^{3\cdot 2^{m-\lambda-1}}_{(\lambda)\iota})}$.

Each triple depends upon all the preceding triples; these being known, two points of each triple determine (in general uniquely) the third point.

The preceding general proposition, successively applied, shows that: Whatever perspective relations among the six points $K_1 \ldots . K_6$ hold in the simple plane Π hold likewise in the plane $\Pi_{C^2(H_{(0)\iota})}$, and hence in the plane $\Pi_{C^4(H^2_{(0)\iota} H_{(1)\iota})}$ and hence in the plane $\Pi_{C^8(H^4_{(0)\iota} H^2_{(1)\iota} H_{(2)\iota})}$, and so on and in general in the plane

$$\Pi_{C^{2^{m+1}}\underset{\lambda=0.1\ldots.m}{(H^{2^{m-\lambda}}_{(\lambda)\iota})}}.*$$

Conversely, if a perspective relation among the six points $K_1 \ldots . K_6$ holds in the plane with any one of these nets of curves $\Pi_{C^{2^{m+1}}}\underset{\lambda=0.1\ldots.m}{(H^{2m-\lambda}_{(\lambda)\iota})}$, then it holds in all, and so also in the simple plane Π.

YALE, NEW HAVEN, CONN., March 20, 1888.

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*In general case, for comparison of the theorems, set $H_{(\lambda)\iota} \equiv G_{(m-\lambda)\iota}$.